Rank 1 Facets of the Set Packing Polyhedron as Limited Memory Cuts

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Set Packing Problem

Formulation

$$\min \sum_{j=1}^{m} c_j x_j$$

$$\text{s.t.} \quad Ax \leq \mathbb{1},$$

$$x \in \{0,1\}^m.$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Set Packing Problem

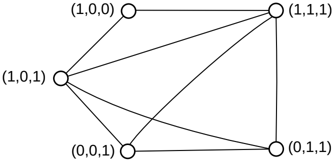


Figure 1: Intersection graph

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Set Packing Polytope

The set packing polytope associated with a matrix A is:

$$SPP_{\leq}(A) = \text{Conv}\left\{Ax \leq \mathbb{1}, x \in \{0,1\}^m\right\}$$

Also denoted as $SPP_{\leq}(G)$, where G is the intersection graph derived from A

Set Packing Polytope

$SPP_{\leq}(G)$ is a classical polytope, studied since the seventies.

- Facet-inducing inequalities associated with specific graphs:
  - cliques (Padberg, 1973)
  - odd holes (Padberg, 1973)
  - odd anti-holes (Nemhauser and Trotter, 1974)
  - webs and anti-webs (Trotter, 1975)
  - $K_{1,3}$-free graphs (Giles and Trotter, 1981)
  - wheels (Cheng and Cunningham, 1997)
  - antiweb-wheels (Cheng and Vries, 2002)
  - grilles (Cánovas et al., 2000).

Set Packing Polytope

- Complete characterizations of $SPP_{\leq}(G)$ are known for:
  - perfect graphs (Chvátal, 1975).
  - series-parallel graphs (Mahjoub, 1988; Barahona and Mahjoub, 1994a).
  - graphs that do not have a 4-wheel as a minor (Barahona and Mahjoub, 1994b).

Set Packing Polytope

- Facet-generating procedures (lifting, composition of graphs, and others) for $SPP_{\leq}(G)$ were described by:
  - Nemhauser and Trotter (1974).
  - Chvátal (1975).
  - Wolsey (1976).
  - Padberg (1977).
  - Balas and Zemel (1977).
  - Barahona and Mahjoub (1994a).
  - Cánovas et al. (2000).
  - Rossi and Smriglio (2001).
  - Rebennack et al. (2011).
  - S. Xavier and Campêlo (2011).
  - Corrêa et al. (2017).

Set Packing Cuts in Column Generation

- Many applications can be modeled as set packing/partitioning problems with a very large number of columns.

- Explicit representation of the coefficient matrix is unpractical.

- Any cutting plane should have a well-defined coefficient for every possible column.

Complete Set Packing Polytope

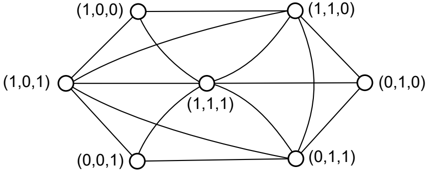
Complete set packing polytope:

$$CSPP_{\leq}(n) = \text{Conv}\left\{\sum_{j=1}^{2^n-1} b^j \lambda_j \leq \mathbb{1}, \lambda \in \{0,1\}^{2^n-1}\right\}$$

where b^j is the column associated with the binary representation of j.

For example, if $n = 3$ we have:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



Why not partitioning?

- For several problems, one could study the set partitioning polytope to derive cuts.
  - Capacitated Vehicle Routing Problem
  - Heterogeneous Fleet Vehicle Routing Problem

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- For several problems, one could study the set partitioning polytope to derive cuts.
  - Capacitated Vehicle Routing Problem
  - Heterogeneous Fleet Vehicle Routing Problem
- As shown in (Bulhões et al., 2018), there is no significant difference between the complete set packing and the complete set partitioning polytopes!

Why not partitioning?

Theorem 1 (Bulhões et al., 2018)

Let

$$\sum_{v \in K} \lambda_v = 1, \forall K \in \mathcal{K}_=, \tag{0.1}$$

$$\sum_{v \in K} \lambda_v \leq 1, \forall K \in \mathcal{K}_\leq, \tag{0.2}$$

$$H\lambda \leq h, \tag{0.3}$$

$$\lambda_j \geq 0, \forall j \in \{1, \ldots, 2^n - 1\} \setminus \{2^0, \ldots, 2^{n-1}\}, \tag{0.4}$$

be a minimal description of $CSPP_=(n)$ *where inequalities* (0.3) *are of the form described in Lemma 3. Then,*

$$\sum_{v \in K} \lambda_v \leq 1, \forall K \in \mathcal{K}_= \cup \mathcal{K}_\leq, \tag{0.5}$$

$$H\lambda \leq h, \tag{0.6}$$

$$\lambda_j \geq 0, \forall j \in \{1, \ldots, 2^n - 1\}, \tag{0.7}$$

is a minimal description of $CSPP_\leq(n)$.

Rank 1 Cuts

Rank 1 Cuts

$$
\begin{aligned}
0\lambda_1 + 1\lambda_2 + 0\lambda_3 + 1\lambda_4 + 0\lambda_5 &\leq 1\\
1\lambda_1 + 1\lambda_2 + 1\lambda_3 + 0\lambda_4 + 0\lambda_5 &\leq 1\\
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\lambda \text{ integer}
\end{aligned}
$$

multipliers vector $u = (0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, 0)^\top$

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Rank 1 Cuts

$$0\left(0\lambda_1 + 1\lambda_2 + 0\lambda_3 + 1\lambda_4 + 0\lambda_5 \leq 1\right)$$
$$\frac{1}{2}\left(1\lambda_1 + 1\lambda_2 + 1\lambda_3 + 0\lambda_4 + 0\lambda_5 \leq 1\right)$$
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$$\lambda \text{ integer}$$

Summing up:

$$1.5\lambda_1 + \lambda_2 + 0.5(\lambda_3 + \lambda_4 + \lambda_5) \leq 1.5$$

Rank 1 Cuts

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Rounding down LHS:

$$\lambda_1 + \lambda_2 \le 1.5$$

Rounding down RHS:

$$\lambda_1 + \lambda_2 \le 1$$

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Subset Row Cuts

- R1Cs with $u = (\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$
- Examples:
  - $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$
  - $(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3})$
- Used in (Jepsen et al., 2008), (Baldacci et al., 2011) and (Contardo and Martinelli, 2014)
- Potentially very effective

The Breakthrough

Pecin et al., 2014 proposed the the limited memory technique for greatly reducing the impact of SRCs in the pricing and could obtain the full benefit of those cuts.

- In CVRP: from 150 to 360 customers!

Diego Pecin, Artur Pessoa, Marcus Poggi, and Eduardo Uchoa. Improved branch-and-cut-and-price for capacitated vehicle routing. In Integer Programming and Combinatorial Optimization, pages 393-403, Springer, 2014.

Finding the best multipliers

What are the best multipliers?

Computational study of Pecin et al. (2017) for $n \leq 5$.

- $n = 3: \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
- $n = 4: \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
- $n = 5: \left(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{2}{4}, \frac{2}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right),$
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What are the best multipliers?

Theoretical study of Bulhões et al. (2018) for arbitrary n.

Theorem 2

The following multipliers induce rank 1 facets of $CSPP_{\leq}(n)$.

(a) $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right), n \geq 4$

(b) $\left(\frac{n-3}{n-1}, \frac{2}{n-1}, \frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right), n \geq 5$

(c) $\left(\frac{n-2}{n}, \frac{2}{n}, \frac{2}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right), n \geq 5$

(d) $\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, *if n is odd*

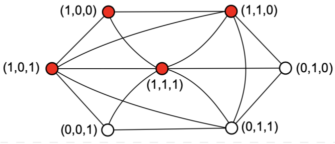
(e) $\left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)$, *if $n = 3k + 2, k \in \mathbb{N}, k \geq 1$*

(f) $\left(\frac{n-3}{n-2}, \frac{n-3}{n-2}, \frac{2}{n-2}, \frac{1}{n-2}, \ldots, \frac{1}{n-2}\right), n \geq 5$

(g) $\left(\frac{n-2}{n-1}, \frac{n-2}{n-1}, \frac{2}{n-1}, \frac{2}{n-1}, \frac{1}{n-1}, \ldots\right), n \geq 5$

What are the best multipliers?

- Clique cuts



$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

What are the best multipliers?

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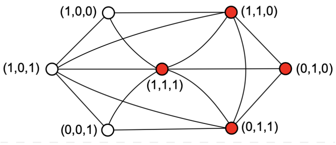

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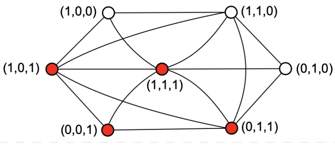

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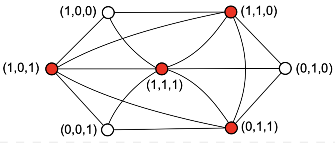

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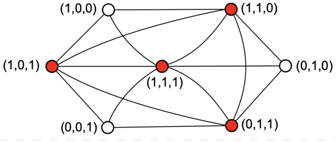

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The corresponding cut

$$x_3 + x_5 + x_6 + x_7 \leq 1$$

is a R1C with $u = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$.

What are the best multipliers?

Theoretical study of Bulhões et al. (2018) for arbitrary n

Lemma 1

Any set of multipliers $u = (\frac{\alpha_1}{\beta}, \frac{\alpha_2}{\beta}, \ldots, \frac{\alpha_n}{\beta})$, with $\alpha_1, \alpha_2, \ldots, \alpha_n$ and β integers, such that $\sum_{i=1}^{n} \alpha_i = 2\beta - 1$ induces a clique inequality.

Lemma 2

Every rank 1 clique inequality $a^T \lambda \leq 1$ can be generated by multipliers $u = (\frac{\alpha_1}{\beta}, \frac{\alpha_2}{\beta}, \ldots, \frac{\alpha_n}{\beta})$, with $\alpha_1, \alpha_2, \ldots, \alpha_n$ and β integers, such that $\sum_{i=1}^{n} \alpha_i = 2\beta - 1$.

Limited Memory Rank 1 Cuts in VRPSolver

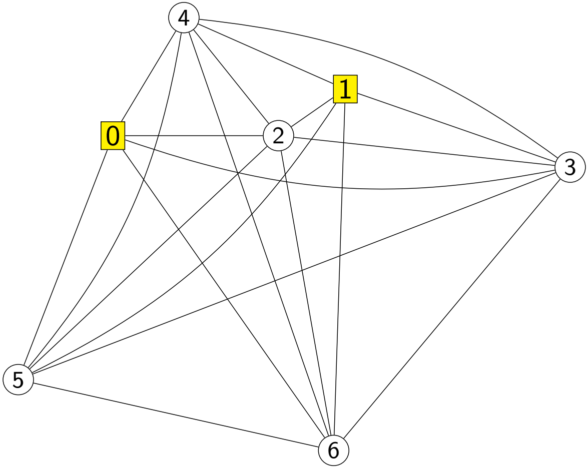
Packing sets concept

Let $\mathcal{B}^{\mathcal{V}} \subset 2^V$ be a collection of mutually disjoint subsets of V such that the constraints:

$$\sum_{p \in P} \left(\sum_{v \in B} h_v^p \right) \lambda_p \leq 1, \quad B \in \mathcal{B}^{\mathcal{V}},$$

are satisfied by at least one optimal solution. In those conditions, we say that the elements of $\mathcal{B}^{\mathcal{V}}$ are *packing sets on vertices*.

Multi-Depot VRP



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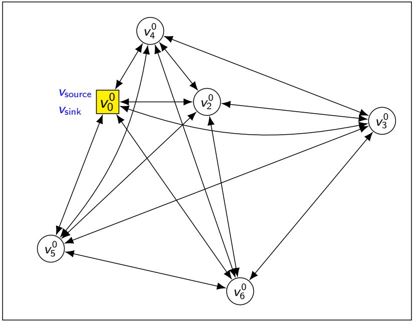
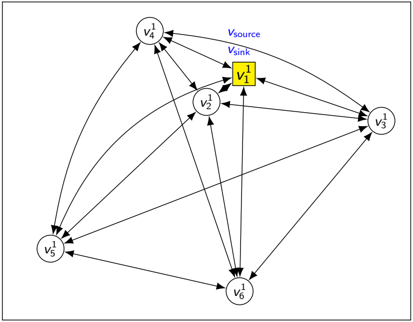


Graph G^0



Graph G^1

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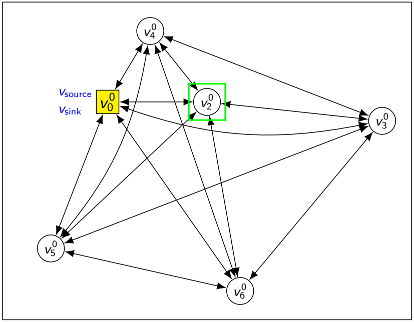
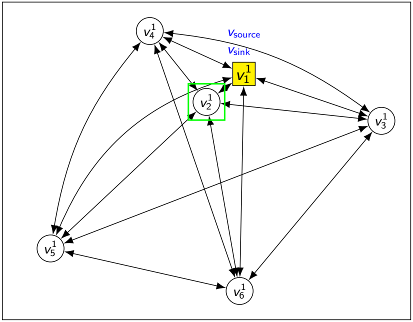


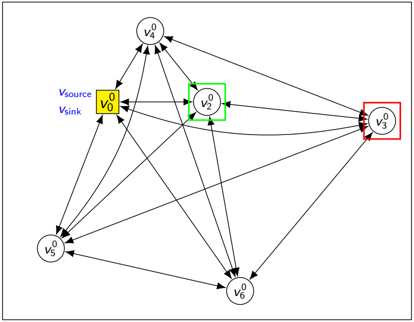
Graph G^0



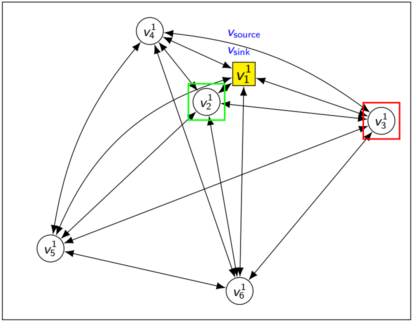
Graph G^1

$\mathcal{B}^{\mathcal{V}} = \{ \{v_2^0, v_2^1\}$

Multi-Depot VRP



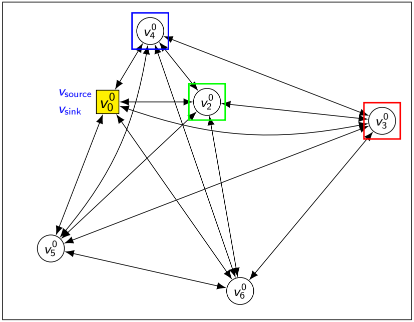
Graph G^0



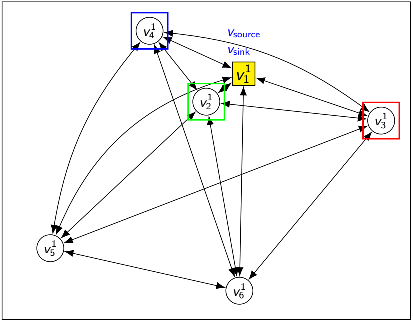
Graph G^1

$\mathcal{B}^{\mathcal{V}} = \{ \{v_2^0, v_2^1\}, \{v_3^0, v_3^1\} \}$

Multi-Depot VRP



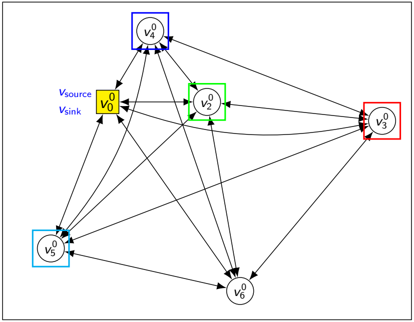
Graph G^0



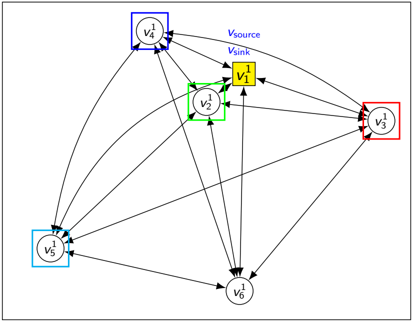
Graph G^1

$\mathcal{B}^{\mathcal{V}} = \{ \{v_2^0, v_2^1\}, \{v_3^0, v_3^1\}, \{v_4^0, v_4^1\}$

Multi-Depot VRP



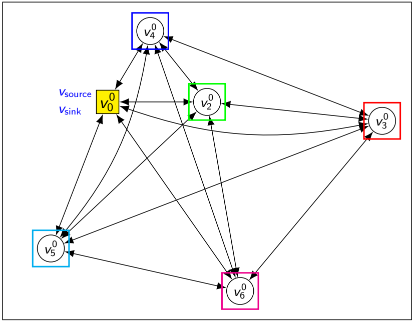
Graph G^0



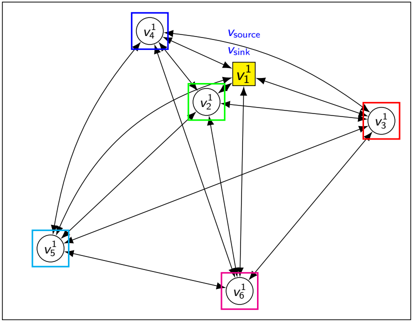
Graph G^1

$\mathcal{B}^{\mathcal{V}} = \{ \{v_2^0, v_2^1\}, \{v_3^0, v_3^1\}, \{v_4^0, v_4^1\}, \{v_5^0, v_5^1\}$

Multi-Depot VRP



Graph G^0



Graph G^1

$$\mathcal{B}^{\mathcal{V}} = \{ \{v_2^0, v_2^1\}, \{v_3^0, v_3^1\}, \{v_4^0, v_4^1\}, \{v_5^0, v_5^1\}, \{v_6^0, v_6^1\} \}$$

Packing sets: Examples

Capacitated Vehicle Routing Problem:

$$\mathcal{B}^{\mathcal{V}} = \cup_{i \in V_+}\{\{i\}\}$$

Heterogeneous Fleet Vehicle Routing Problem:

$$\mathcal{B}^{\mathcal{V}} = \cup_{i \in V_+}\{\{v_i^k : k \in K\}\}$$

Team Orienteering Problem:

$$\mathcal{B}^{\mathcal{V}} = \cup_{i \in V_+}\{\{i\}\}$$

Concluding remarks

- The limited memory Rank 1 Cuts are a key feature of state-of-the-art exact solvers for several VRPs
- VRPSolver will automatically add them using the packing sets defined by the user
- Future works on these cuts include:
  - Investigating better separation algorithms
  - Investigating the impact of the newly proposed multipliers

Thank you for your attention!

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