Branch-Cut-and-Price for the Robust Capacitated Vehicle Routing Problem under Knapsack Uncertainty

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Outline

1. Definition and formulation
2. The pricing problem
3. Experiments
4. OJMO - Open Journal of Mathematical Optimization

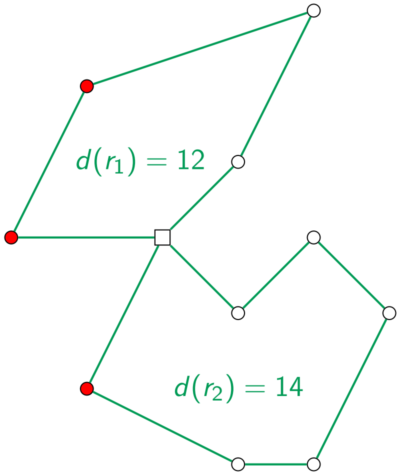
Deterministic problem definition

Instance description

- V nodes
- A arcs
- 0 depot
- K identical vehicles
- C Capacity
- d demand
- c cost

Example

- $K = 2,\ C = 16$
- $d_i = 2$
- $d_i = 4$



Partition-constrained uncertainty

$\mathcal{D}_{part}$ [Gounaris et al., 2013]

Partition: $V = V_1 \cup \cdots \cup V_s$

$$\left\{ d \in [\overline{d}, \overline{d} + \hat{d}] \mid \sum_{i \in V_j} d_i \leq b_j, j \in S \right\}$$

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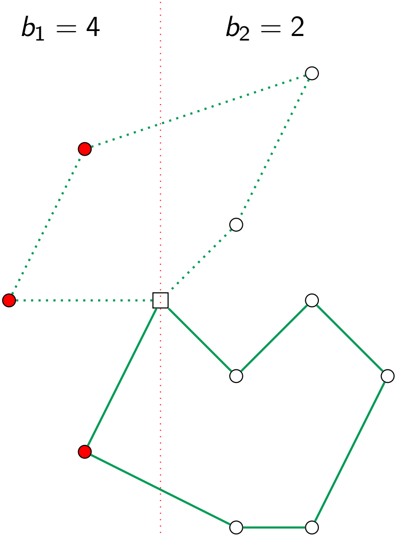
- $K = 2, \ C = 16$
- $\overline{d}_i = 2, \ \hat{d}_i = 1$
- $\overline{d}_i = 4, \ \hat{d}_i = 2$



$d^*(r) = 6 + 12 = 18 > C$

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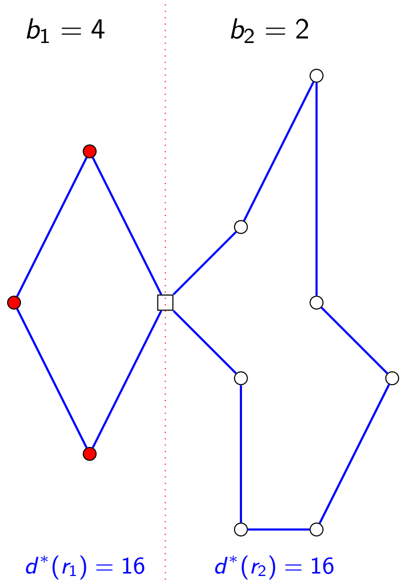

Cardinality-constrained uncertainty

$\mathcal{D}_{card}$ [Bertsimas and Sim, 2003]

$$\left\{ d \in [\overline{d}, \overline{d} + \hat{d}] \mid \sum_{i \in V} \frac{d_i - \overline{d}_i}{\hat{d}_i} \le \Gamma \right\}$$

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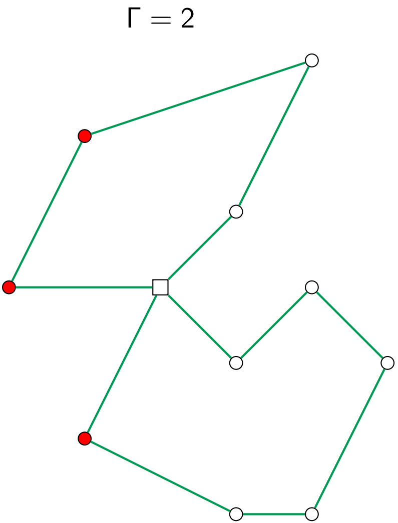

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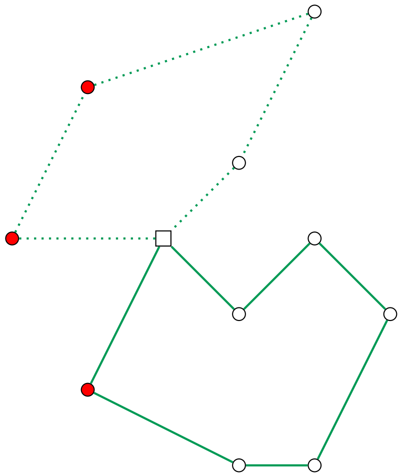

$d^*(r) = 4 + 2 + 10 + 1 = 17 > C$

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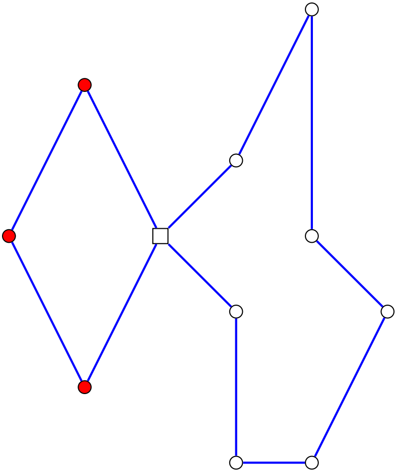

$d^*(r_1) = 16$ $\quad$ $d^*(r_2) = 16$

We consider the general model

$$\mathcal{D} \equiv \left\{ d \in [\overline{d}, \overline{d} + \hat{d}] \mid \sum_{i \in V_k} w_i d_i \leq b_k,\ k = 1, \ldots, s \right\},$$

with $V = V_1 \cup \cdots \cup V_s$ and $V_k \cap V_\ell = \emptyset$ for all $k \neq \ell$.

Set partitioning (master) formulation

Robust-feasible routes

$$R(\mathcal{D}) = \left\{ r \in R_0 \;\middle|\; \sum_{i \in r} \alpha_i^r d_i \leq C, \quad \forall d \in \mathcal{D} \right\}.$$

Notations

- α_i^r number of times vertex i appears in route r.
- c_r cost of route r.
- λ_r $= 1$ iff route $r \in R$ is used.

$$\begin{aligned} \min \quad & \sum_{r \in R(\mathcal{D})} c_r \lambda_r, \\ \text{s.t.} \quad & \sum_{r \in R(\mathcal{D})} \alpha_i^r \lambda_r = 1, \quad i \in V_0, \\ & \sum_{r \in R(\mathcal{D})} \lambda_r = K, \\ & \lambda_r \in \{0,1\}, \quad r \in R(\mathcal{D}). \end{aligned}$$

Note: $R(\mathcal{D})$ can be enlarged to include non-elementary routes.

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Pricing problem

Notations

- κ reduced costs
- C capacity
- x arcs in the route
- $\mathcal{X}$ set of routes

$$\min \sum_{(i,j)\in A} \kappa_{ij} x_{ij},$$

$$\text{s.t.} \sum_{(i,j)\in A} d_j x_{ij} \leq C, \quad d \in \mathcal{D}$$

$$x \in \mathcal{X}$$

Question

How to solve the pricing problem efficiently?

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Simplification: $\mathcal{D} \equiv \left\{ d \in [0, \hat{d}] \mid \sum_{i \in V_j} d_i \leq b_j, j \in S \right\}$

$$\sum_{(i,j)\in A} d_j x_{ij} \leq C, \quad d \in \mathcal{D} \quad \Leftrightarrow \quad \max \left\{ \sum_{(i,j)\in A} \boldsymbol{d}_j x_{ij} \,\middle|\, \boldsymbol{d} \in \mathcal{D} \right\} \leq C$$

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Remark: All kink-points of $f_\chi(\boldsymbol{\theta})$ are in $\{0, 1\}^{|S|}$

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Pricing problem for the partition-constrained uncertainty

Theorem

The problem

$$\min_{x\in\mathcal{X}}\left\{\sum_{(i,j)\in A}\kappa_{ij}x_{ij} : \sum_{(i,j)\in A} d_j x_{ij} \leq C, \quad d\in\mathcal{D}\right\}$$

amounts to solve $2^{|S|} = |\Theta|$ *problems*

$$\min_{x\in\mathcal{X}}\left\{\sum_{(i,j)\in A}\kappa_{ij}x_{ij} : \sum_{(i,j)\in A} d_j^{\theta} x_{ij} \leq C^{\theta}\right\}$$

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Can we reduce the number of problems?

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Reducing the number of problems

Reminder: $\mathcal{D} \equiv \left\{ d \in [\overline{d}, \overline{d} + \hat{d}] \mid \sum_{i \in V_k} w_i d_i \leq b_k,\ k = 1, \dots, s \right\}$

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The pricing amounts to solve many problems of the type

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Proof.

Using complementary slackness conditions ... □

- Changes the problem structure ☹
- Use these new constraints to test feasibility of (P^θ)
- (P^θ) feasible $\Rightarrow$

$$\min_{x \in \{0,1\}^n} \left\{ \sum_{(i,j)\in A} d_j^\theta x_{ij} : \tilde{w}^T x \geq b \right\} \leq C^\theta$$

(*notice* $\mathcal{X}$ *has been relaxed to* $\{0,1\}^n$)

Example ($\mathcal{D}_{card}$)

$\tilde{w} = 1$ so the feasibility check becomes

$$\min_{x \in \{0,1\}^n} \left\{ \sum_{(i,j)\in A} d_j^\theta x_{ij} : \sum_i x_i \geq \Gamma \right\}.$$

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Reminder: $\mathcal{D} \equiv \left\{ d \in [\overline{d}, \overline{d} + \hat{d}] \mid \sum_{i \in V_k} w_i d_i \leq b_k,\ k = 1, \ldots, s \right\}$

Theorem

The pricing amounts to solve many problems of the type

$$
\begin{aligned}
(P^\theta) \quad & \min \sum_{(i,j)\in A} \kappa_{ij} x_{ij}, \\
& \text{s.t.} \sum_{(i,j)\in A} d_j^\theta x_{ij} \leq C^\theta \\
& \tilde{w}^T x \geq b \\
& x \in \mathcal{X}
\end{aligned}
$$

Proof.

Using complementary slackness conditions ... □

- Changes the problem structure ☹
- Use these new constraints to test feasibility of (P^θ)
- (P^θ) feasible $\Rightarrow$

$$\min_{x \in \{0,1\}^n} \left\{ \sum_{(i,j)\in A} d_j^\theta x_{ij} : \tilde{w}^T x \geq b \right\} \leq C^\theta$$

(*notice* $\mathcal{X}$ *has been relaxed to* $\{0,1\}^n$)

Example ($\mathcal{D}_{card}$)

$\tilde{w} = 1$ so the feasibility check becomes

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Reformulation

- Necessary kinkpoints: $\Theta^* \equiv \{\theta \in \Theta : P^\theta \neq \emptyset\}$
- Robust-feasible routes: $R_\theta = \left\{ r \in R_0 \;\middle|\; \sum_{i \in r} \alpha_i^r d_i^\theta \leq C_\theta, \quad \forall d \in \mathcal{D} \right\}.$

Parameters

- α_i^r number of times vertex i appears in route r.
- c_r cost of route r.
- λ_r $= 1$ iff route $r \in R$ is used.
- Θ^* set of vehicles types

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\begin{aligned}
\min \quad & \sum_{\theta \in \Theta^*} \sum_{r \in R_\theta} c_r \lambda_r, \\
\text{s.t.} \quad & \sum_{\theta \in \Theta^*} \sum_{r \in R_\theta} \alpha_i^r \lambda_r = 1, \quad i \in V_0, \\
& \sum_{\theta \in \Theta^*} \sum_{r \in R_\theta} \lambda_r = K, \\
& \lambda_r \in \{0,1\}, \quad \theta \in \Theta^* r \in R_\theta.
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robust homogeneous CVRP $\approx$ deterministic heterogeneous CVRP

$\Rightarrow$ we can use VRPSolver

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$$

robust homogeneous CVRP $\approx$ deterministic heterogeneous CVRP

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Effect of preprocessing for $\mathcal{D}_{part}$

In. cls	#in.	K	#sp	%red.	#det.
A	26	7.1	2.7	83.4%	7
B	23	7.2	3.9	75.8%	0
E	11	7.3	3.6	77.3%	4
F	3	5.0	5.0	68.8%	0
M	3	9.7	1.3	91.7%	2
P	24	7.3	4.3	73.2%	9
all	90	7.2	3.6	77.8%	22

22 out of 90 instances were reduced to deterministic ones

Outline

Branch-cut-and-price algorithm

Uses BaPCod, which is state-of-the-art for several VRP variants [Pessoa et al., 2019]

Features:

- Bidirectional bucket-graph based pricing with bucket-arc fixing (allows fractional resource consumption)
  - Multilevel pricing heuristics
  - Ng-path for partial elimination of cycles [Baldacci et al., 2011]
- Automatically parametrized stabilization [Pessoa et al., 2018]
- Rank-1 cuts over the λ variables with limited memory [Pecin et al., 2017]
- Multilevel strong branching with history information
- Enumeration of useful elementary routes when the gap is sufficiently small [Baldacci and Mingozzi, 2009]

Experiments setup

- Intel(R) Core(TM) i7-3770 machine with 12 Gb of RAM
- geometric means are used for all run times
- For $\mathcal{D}_{part}$, instances from Gounaris et al. [2013]
  - 90 instances with 15 to 150 customers
  - Mean demands $= 0.9\times$ Original demands
  - Deviations $= 0.2\times$ Original demands
  - Capacities $= 1.2\times$ Original capacities
  - 4 uncertainty partitions associated to quadrants
  - Deviation sum limits $= 0.15\times$ total demands in partitions
- Literature results for $\mathcal{D}_{part}$
  - Exact method: BC by Gounaris et al. [2013]
  - Heuristic: AMP by Gounaris et al. [2016] (1h run)
  - AMP+BC for unsolved instances:
    5 min of AMP + 24h of BC (we use BC times for solved)
- For $\mathcal{D}_{card}$, new instances

Valid inequalities effect for $\mathcal{D}_{part}$ (NOT PRESENTED)

In. cls	# in.	BCP root gap 0	t. 0	#c.r.	gap 1	t. 1
A	26	2.16%	0.70	3.7	0.00%	2.91
B	23	3.68%	1.31	2.8	0.01%	5.95
E	11	2.31%	2.79	5.4	0.00%	11.40
F	3	3.01%	139.10	5.0	0.30%	309.40
M	3	1.66%	12.49	14.7	0.20%	52.44
P	24	1.27%	0.51	2.8	0.00%	1.47
all	90	2.34%	1.17	3.9	0.02%	4.43

Branch-cut-and-price results for $\mathcal{D}_{part}$

In. cls	# in.	BCP #n.	BCP t.	BCP #opt.	AMP+BC gap	AMP+BC t.	AMP+BC #opt.
A	26	1.00	2.91	26	1.97%	3440.31	12
B	23	1.05	5.98	23	1.39%	250.96	13
E	11	1.00	11.40	11	2.19%	573.01	5
F	3	5.37	833.42	2	1.10%	55.76	2
M	3	3.33	153.51	3	2.70%	86700.00	1
P	24	1.00	1.48	24	2.09%	976.36	10
all	90	1.11	4.75	89	1.87%	981.90	43

Conclusion

Take-away message

$$\min_{x \in X} \left\{ c^\top x : a^\top x \le C, \quad \forall a \in \mathcal{D} \right\} \quad = \quad \min_{\theta \in \Theta} \min_{x \in X} \left\{ c^\top x : a_\theta{}^\top x \le C_\theta \right\}$$

- Useful in decomposition algorithms (pricing problem)
- The number of subproblems can be reduced substantially

$\Rightarrow$ Applications in scheduling, bin-packing, ...

Other contributions (NOT PRESENTED)

- Reinforced capacity inequalities
- Improved heuristics

A. A. Pessoa, M. Poss, R. Sadykov, F. Vanderbeck. Branch-and-cut-and-price for the robust capacitated vehicle routing problem with knapsack uncertainty. Minor revision pending for Operations Research

Outline

Open Journal of Mathematical Optimization (OJMO)

- Where will you publish your new improvements on VRPSolver?
- With classical publishers, your own institution will have to pay dearly to access it, unless you pay ($\pm$2k) for "gold" Open Access
  ...and your institution will pay the same to access all the others
- OJMO provides a free OA alternative (thanks to Mersenne)
  https://ojmo.centre-mersenne.org/
- Papers will have DOI, indexation will follow
- We seek high-quality contributions

Steering Commitee

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- Martine Labbé
- Eva K. Lee
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Area Editors

- Continuous Optimization – David Russell Luke
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